

RECTANGULAR QUASIGROUPS AND RECTANGULAR LOOPS

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ABSTRACT. We solve two problems posed by Krapež by finding a basis of seven independent axioms for the variety of rectangular loops. Six of these axioms form a basis for the variety of rectangular quasigroups. The proofs of the lemmas showing that the six axioms are sufficient are based on proofs generated by the automated reasoning program OTTER, while most of the models verifying the independence of the axioms were generated by the finite model builder Mace4.

1. INTRODUCTION

A *rectangular band* is a semigroup $\mathcal{B} = (B; \cdot)$, which is a direct product $\mathcal{B} = \mathcal{L} \oplus \mathcal{R}$ of a left zero semigroup $\mathcal{L} = (L; \cdot)$ (that is, a semigroup satisfying $x \cdot y = x$) and a right zero semigroup $\mathcal{R} = (R; \cdot)$ (that is, a semigroup satisfying $x \cdot y = y$). A semigroup that is a direct product of a group and a rectangular band is called a *rectangular group*.

In a series of papers [1, 2, 3, 4], Krapež has generalized the notion of rectangular group to rectangular quasigroup and rectangular loop. Recall that a *quasigroup* $\mathcal{Q} = (Q; \cdot)$ is a set Q together with a binary operation $\cdot : Q \times Q \rightarrow Q$ such that for each $a \in Q$, the mappings $x \mapsto a \cdot x$ and $x \mapsto x \cdot a$ are bijections. This induces two other binary operations $\backslash, / : Q \times Q \rightarrow Q$ as follows. For $x, y \in Q$, $x \backslash y$ is the unique solution u to the equation $x \cdot u = y$ and x / y is the unique solution v to the equation $v \cdot y = x$. This leads to an equivalent definition: a *quasigroup* $\mathcal{Q} = (Q; \cdot, \backslash, /)$ is a set Q with three binary operations $\cdot, \backslash, / : Q \times Q \rightarrow Q$ satisfying the equations:

$$\begin{aligned} x \backslash (x \cdot y) &= y & (x \cdot y) / y &= x \\ x \cdot (x \backslash y) &= y & (x / y) \cdot y &= x \end{aligned}$$

A *loop* is usually defined as a quasigroup with a neutral element $1 \in Q$ satisfying $1 \cdot x = x$ and $x \cdot 1 = x$. Loops can also be characterized equationally as quasigroups satisfying the additional axiom $x \backslash x = y / y$. Basic references for quasigroup and loop theory are [5, 6, 7, 8].

Following Krapež, a *rectangular quasigroup* is a direct product of a quasigroup and a rectangular band, while a *rectangular loop* is a direct product of a loop and a rectangular band. Both can be viewed as varieties of algebras with three binary operations $\cdot, \backslash, /$. To consider a rectangular band as an algebra with three operations, one simply sets $x \backslash y = x / y = x \cdot y$. (This is a convention; different choices lead to different axioms.)

In [1], Krapež found a set of 12 axioms characterizing rectangular loops as algebras $\mathcal{M} = (M; \cdot, \backslash, /)$. He indicated that he had shown that some of the axioms are independent, but believed that not all of them are. He also posed the problem of finding an independent set of axioms ([1], Problem 1, p. 66). In [2], he found a set of 15 axioms characterizing rectangular quasigroups, and again posed the problem of finding an independent set of axioms. The two axiom systems are quite distinct, that is, none of Krapež's axioms for rectangular quasigroups occur as axioms for rectangular loops.

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In this paper, we solve both of the posed problems. We give a system of six axioms and show that this is sufficient to characterize the variety of rectangular quasigroups. We then show that these same six plus one additional axiom characterize the variety of rectangular loops. Finally, we will present models that show that the seven rectangular loop axioms are independent. Here is the statement of our main result.

Theorem 1.1. (1) *The equations*

$$\begin{array}{ll} (Q1) & x \backslash (xx) = x \\ (Q3) & x(x \backslash y) = x \backslash (xy) \\ (Q5) & (x \backslash y) \backslash ((x \backslash y) \cdot zu) = (x \backslash (xz))u \end{array} \quad \begin{array}{ll} (Q2) & (xx)/x = x \\ (Q4) & (x/y)y = (xy)/y \\ (Q6) & (xy \cdot (z/u))/(z/u) = x((yu)/u) \end{array}$$

are an independent system of axioms for the variety of rectangular quasigroups.

(2) *Equations (Q1)-(Q6) together with*

$$(L) \quad x \backslash (x(y \backslash y)) = ((x/x)y)/y$$

are an independent system of axioms for the variety of rectangular loops.

If part (1) of Theorem 1.1 is assumed, then it is easy to show that (L), (Q1)-(Q6) characterize the variety of rectangular loops. Indeed, if $\mathcal{M} = (M; \cdot, \backslash, /)$ satisfies (L), (Q1)-(Q6), then \mathcal{M} is a rectangular quasigroup, and hence a direct product $\mathcal{M} = \mathcal{B} \oplus \mathcal{Q}$ of a rectangular band \mathcal{B} and a quasigroup \mathcal{Q} , each of which satisfies (L). This implies that \mathcal{Q} is a loop, and so \mathcal{M} is a rectangular loop. Conversely, since every rectangular band and every loop satisfy (L), so does every rectangular loop. Thus what remains is to show that (Q1)-(Q6) characterize the variety of rectangular quasigroups (§3) and then to show the independence of (L), (Q1)-(Q6) (§4).

As part of the proof of Theorem 1.1, we will also show that eight of Krapež's fifteen axioms are sufficient to characterize rectangular quasigroups.

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2. OTTER AND MACE4

The proofs of the lemmas in §3 are based on proofs generated by the equational reasoning program OTTER developed by McCune [9]. OTTER can prove theorems from axioms in first-order logic, but is strongest in equational reasoning. Not surprisingly then, most *new* mathematics to come out of automated reasoning has been in fields close to algebra. For general methods for applying automated reasoning to problems in mathematics and other areas, see the book by Wos and Pieper [10]. New results proved by OTTER in particular can be found in the book by McCune and Padmanabhan [11].

It is mathematically sound to use OTTER output directly as the proof of a theorem, as is the practice in [11], for instance. Despite the complexity of OTTER's search procedure, the program can be made to output a *proof object*, which can be independently verified by other software, such as a `lisp` program. However, OTTER's proofs are often long sequences of unintuitive equations, and it is useful to re-express them in a form which a human reader can easily verify. Some discussion of the procedure for "humanizing" proofs occurs in [12]. In loop and quasigroup theory, this was first applied by Kunen [13, 14, 15, 16, 17], and then later by the present authors and Kunen in various combinations of coauthors [18, 19, 20, 21, 22, 23, 24].

The proofs in §3 of the present paper are somewhat closer to the OTTER proofs than in the aforementioned references. In quasigroup and loop theory, one can rely on a great deal of existing machinery (e.g., autotopisms) to simplify proofs. Not so much machinery is available for rectangular quasigroups and rectangular loops. Nevertheless, the proofs herein are still translated into a humanly verifiable form. In particular, we took care to ensure that each step directly uses one of the axioms or a closely related equation, or else uses one of the equations in Proposition 3.1 below.

The models in §4 that show the independence of axioms (Q1)-(Q6) were found using McCune's finite model builder Mace4 [25]. Mace4 can generate its output in a portable form, which can then be used by other programs to independently verify the claimed properties of the models. However, in this case all of the models are quite small, so it was just as easy to verify the properties "by hand".

3. PROOFS

Since every quasigroup and every rectangular band satisfy (Q1)-(Q6), to show that these equations characterize the variety of rectangular quasigroups, it is enough to show that they imply the axioms of Krapež:

Proposition 3.1 ([2]). *The following equations axiomatize the variety of rectangular quasigroups: (Q1)-(Q4) and*

$$(xy) \setminus (xy \cdot z) = x \setminus (xz) \quad (\text{K5})$$

$$(x \setminus y) \setminus ((x \setminus y)z) = x \setminus (xz) \quad (\text{K6})$$

$$(x/y) \setminus ((x/y)z) = x \setminus (xz) \quad (\text{K7})$$

$$x(y \setminus (yz)) = xz \quad (\text{K8})$$

$$((xy)/y)z = xz \quad (\text{K9})$$

$$(x \cdot yz)/(yz) = (xz)/z \quad (\text{K10})$$

$$(x(y \setminus z))/(y \setminus z) = (xz)/z \quad (\text{K11})$$

$$(x(y/z))/(y/z) = (xz)/z \quad (\text{K12})$$

$$x \setminus (x((yz)/z)) = ((x \setminus (xy))z)/z \quad (\text{K13})$$

$$(x \setminus (xy))z = x \setminus (x \cdot yz) \quad (\text{K14})$$

$$x((yz)/z) = (xy \cdot z)/z \quad (\text{K15})$$

Lemma 3.2. $(Q1), (Q3), (K14) \implies$

$$x \setminus (x \cdot xz) = x(x \setminus (xz)) = x \cdot x(x \setminus z) = xz \quad (3.1)$$

Proof. Replace y with x in (K14) and use (Q1) to get $x \setminus (x \cdot xz) = xz$. The other forms of (3.1) follow applying (Q3). \square

Lemma 3.3. (1) $(Q1), (Q3), (K6), (K14) \implies (K5), (K8)$
 (2) $(Q2), (Q4), (K12), (K15) \implies (K10), (K9)$

Proof. For (1): By Lemma 3.2, we may use (3.1). In (K6), replace y with $x \cdot xy$ and use (3.1) to get (K5). Next

$$\begin{aligned} x(y \setminus (yz)) &= x \cdot y(y \setminus z) && \text{by (Q3)} \\ &= x \cdot x(x \setminus [y(y \setminus z)]) && \text{by (3.1)} \\ &= x\{x(x \setminus y) \cdot [x(x \setminus y)] \setminus [y(y \setminus z)]\} && \text{by (K5)} \\ &= x\{[x(x \setminus y)] \setminus [x(x \setminus y) \cdot y(y \setminus z)]\} && \text{by (Q3)} \\ &= x\{[x(x \setminus y)] \setminus [(x \setminus (xy)) \cdot y(y \setminus z)]\} && \text{by (Q3)} \\ &= x\{[x(x \setminus y)] \setminus [x \setminus (x(y \cdot y(y \setminus z)))]\} && \text{by (K14)} \\ &= x\{[x(x \setminus y)] \setminus [x \setminus (x \cdot yz)]\} && \text{by (3.1)} \\ &= x\{[x(x \setminus y)] \setminus [(x \setminus (xy))z]\} && \text{by (K14)} \\ &= x\{[x(x \setminus y)] \setminus [x(x \setminus y) \cdot z]\} && \text{by (Q3)} \\ &= x\{x(x \setminus y) \cdot [(x \setminus (xy)) \setminus z]\} && \text{by (Q3)} \\ &= x \cdot x(x \setminus z) && \text{by (K5)} \\ &= xz && \text{by (3.1)} \end{aligned}$$

This is (K8).

The proof of (2) is the mirror of that of (1). □

Lemma 3.4. (1) $(Q_4), (K5), (K9) \implies (K7)$
 (2) $(Q_3), (K10), (K8) \implies (K11)$

Proof. For (1):

$$\begin{aligned}
 (x/y) \setminus [(x/y)z] &= [(x/y)y] \setminus [(x/y)y \cdot z] && \text{by (K5)} \\
 &= [(x/y)y \cdot z] \setminus [((x/y)y \cdot z)z] && \text{by (K5)} \\
 &= [((xy)/y)z] \setminus [((xy)/y)z \cdot z] && \text{by (Q4) twice} \\
 &= (xz) \setminus (xz \cdot z) && \text{by (K9) twice} \\
 &= x \setminus (xz) && \text{by (K5)}
 \end{aligned}$$

which establishes (K7).

The proof of (2) is the mirror of that of (1). □

Lemma 3.5. $(Q1), (Q3), (K14), (K15) \implies (K13)$.

Proof. By Lemma 3.2, we may use (3.1). We compute

$$\begin{aligned}
 [(x \setminus (xy))z]/z &= [x(x \setminus y) \cdot z]/z && \text{by (Q3)} \\
 &= x[(x \setminus y)z]/z && \text{by (K15)} \\
 &= x \setminus (x\{x[(x \setminus y)z]/z\}) && \text{by (3.1)} \\
 &= x \setminus (x\{[x(x \setminus y) \cdot z]/z\}) && \text{by (K15)} \\
 &= x \setminus [(x \cdot x(x \setminus y))z]/z && \text{by (K15)} \\
 &= x \setminus [(xy \cdot z)/z] && \text{by (3.1)} \\
 &= x \setminus (x[(yz)/z]) && \text{by (K15)}
 \end{aligned}$$

and this is (K13). □

Combining Proposition 3.1 and the lemmas, we obtain the following.

Theorem 3.6. *Krapež's axioms (Q1)-(Q4), (K6), (K12), (K14), (K15) are sufficient to characterize the variety of rectangular quasigroups.*

So to complete the characterization part of the proof of Theorem 1.1, we need only the following.

Lemma 3.7. (1) $(Q1), (Q3), (Q5) \implies (K6), (K14)$.
 (2) $(Q2), (Q4), (Q6) \implies (K12), (K15)$.

Proof. For (1): First, we compute

$$\begin{aligned}
 (x \setminus y) \setminus [(x \setminus y)z] &= (x \setminus y) \setminus [(x \setminus y)(z \setminus (zz))] && \text{by (Q1)} \\
 &= (x \setminus y) \setminus [(x \setminus y) \cdot z(z \setminus z)] && \text{by (Q3)} \\
 &= (x \setminus (xz))(z \setminus z) && \text{by (Q5)}
 \end{aligned}$$

so that the expression $(x \setminus y) \setminus [(x \setminus y)z]$ is constant in y , that is,

$$(x \setminus y) \setminus [(x \setminus y)z] = (x \setminus u) \setminus [(x \setminus u)z].$$

Take $u = xx$ and apply (Q1) to get (K6). Next,

$$\begin{aligned}
 x \setminus (x \cdot yz) &= (x \setminus u) \setminus [(x \setminus u) \cdot yz] && \text{by (K6)} \\
 &= x \setminus (xy \cdot z) && \text{by (Q5)}
 \end{aligned}$$

and this is (K14).

The proof of (2) is the mirror of that of (1). □

4. INDEPENDENCE OF THE AXIOMS

In this section, we present models that show the independence of axioms (L), (Q1)-(Q6), and this will complete the proof of the Theorem. Note that equation (L) is equivalent to its own mirror, while the other axioms come in mirrored pairs. Thus once we have presented a model satisfying, for instance, all axioms except (Q6), it follows that the same underlying set with the dual operations $x \odot y := y \cdot x$, $x \backslash y := y/x$, $x//y := y \backslash x$ will be a model satisfying all axioms except (Q5). Thus it is enough to present four models.

The independence of (L) is obvious, because any nonloop quasigroup satisfies (Q1)-(Q6), but not (L). Table 1 gives a specific example. Table 2 is a model satisfying (L), (Q1), and (Q3)-(Q6), but not (Q2). Table 3 is a model satisfying (L), (Q1)-(Q3), (Q5)-(Q6), but not (Q4). Table 4 is a model satisfying (L), (Q1)-(Q5), but not (Q6).

\bullet	0	1	2	\backslash	0	1	2	$/$	0	1	2
0	0	1	2	0	0	1	2	0	0	1	2
1	2	0	1	1	1	2	0	1	2	0	1
2	1	2	0	2	2	0	1	2	1	2	0

TABLE 1. (Q1)-(Q6), but not (L)

\bullet	0	1	\backslash	0	1	$/$	0	1
0	0	1	0	0	1	0	1	0
1	1	0	1	1	0	1	0	1

TABLE 2. (L), (Q1), (Q3)-(Q6), but not (Q2)

\bullet	0	1	2	3	\backslash	0	1	2	3	$/$	0	1	2	3
0	0	1	1	0	0	0	1	1	0	0	0	2	1	0
1	0	1	1	0	1	0	1	1	0	1	0	1	1	0
2	3	2	2	3	2	3	2	2	3	2	0	2	2	0
3	3	2	2	3	3	3	2	2	3	3	3	1	1	3

TABLE 3. (L), (Q1)-(Q3), (Q5)-(Q6), but not (Q4)

\bullet	0	1	\backslash	0	1	$/$	0	1
0	0	1	0	0	1	0	0	0
1	0	1	1	0	1	1	0	1

TABLE 4. (L), (Q1)-(Q5), but not (Q6)

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